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3-TREES IN POLYHEDRAL MAPS

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ABSTRACT

We show that the vertices of the graph of any polyhedral map on the projective plane, torus or Klein bottle can be covered by a subgraph that is a tree of maximum valence 3. This extends a theorem of the author, who previously proved this theorem for the graphs of 3-dimensional polytopes. Several theorems dealing with paths in polyhedral maps are a consequence of these theorems.

1. Introduction

In [3] the author proved that if G is a planar 3-connected graph then G has a subgraph T that is a tree, covers the vertices of G, and has maximum valence 3. By a theorem of Steinitz, the planar 3-connected graphs are isomorphic to the graphs consisting of the vertices and edges of convex 3-dimensional polytopes. Natural analogs of planar 3-connected graphs would be the graphs that are embedded in 2-dimensional manifolds so that they induce a facial structure similar to that of 3-dimensional polytopes.

In this paper we extend the above mentioned theorem to these graphs when the manifolds are the projections plane, torus or Klein bottle.

2. Definitions and preliminary results

If a graph G is embedded in a manifold M, then the closures of the connected components of $M - G$ are called the **faces** of G. If each two faces meet on a vertex, an edge or not at all, we say that faces meet **properly**. If each face is a closed 2-cell and faces meet properly we say that G is a **polyhedral map.**

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A graph G is said to be n-**connected** if and only if it has at least $n+1$ vertices and one must remove at least n vertices to disconnect G. By a theorem of Menger [4] this is equivalent to saying that between any two vertices of G there are n paths that meet only their endpoints.

When the faces are 2-cells the condition of faces meeting properly is the same as saying that no two faces have a multiply connected union and all vertices are at least 3-valent. In [2] the author shows that this condition implies 3-connectedness, thus polyhedral maps are 3-connected.

Our method of proof of the main theorem for the Klein bottle or torus will be to take a subset of the faces whose union is an annulus or Möbius strip and contains all of the vertices of M. We will then apply a strengthened version of the above mentioned theorem in [3] to this annulus. In [1] the author proves that such an annular subset always exists.

If T is a tree of maximum valence 3 then we shall call it a **3-tree**. If a subgraph H of G contains all vertices of G we say that H **covers** G.

If G is a planar 3-connected graph embedded in the plane, and if J is a simple (i.e., non-selfintersecting) circuit in G then we denote the graph consisting of J and all vertices and edges of G inside J by $G(J)$. All such graphs will be called **circuit graphs**. An edge of $G(J)$ not on J will be called an **interior edge** of $G(J)$. Any vertex of J that meets an interior edge will be called a **major vertex** of J. A face F of $G(J)$ is said to **separate** $G(J)$ provided there are two vertices of F that separate $G(J)$ into two components, each containing an interior edge.

3. 3-Trees in circuit graphs

In [4] we proved that every circuit graph is covered by a 3-tree. We now strengthen that theorem:

THEOREM 1: *If v is a vertex of a circuit graph $G(J)$ then there is a 3-tree T covering G with v 1-valent in T.*

Proof: Our proof is by induction on the number of interior edges of $G(J)$. The theorem is clearly true if $G(J)$ has no interior edges and is thus a circuit. If $G(J)$ has interior edges, we treat three cases.

CASE I: There is a major vertex $x \neq v$ such that no face meeting x separates J. We apply the inductive step in [3]. This involves removing faces $F_2, \ldots, F_n$ from $G(J)$ as shown in Fig.1.

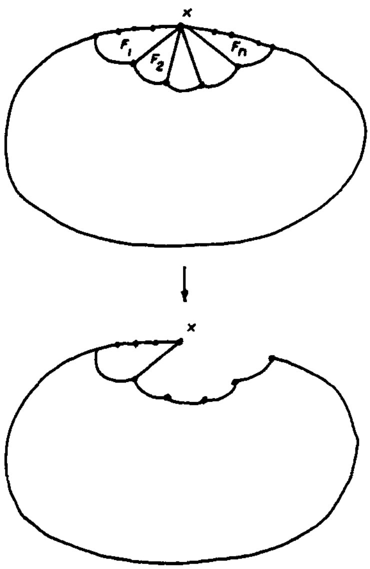

Fig. 1

In the resulting circuit graph $G(J')$, x is 2-valent and by induction there is a 3-tree T' covering $G(J')$ that is at most 2-valent at x. The tree is easily extended to $G(J)$ by adding a path along F_n to cover the 2-valent vertices of F_n. If v is not one of these 2-valent vertices then we may also assume by induction that v is 1-valent in T' and thus also in T. If v is one of these 2-valent vertices, then we apply a symmetric argument, removing the faces $F_1, \ldots, F_{n-1}$ instead of $F_2, \ldots, F_n$. We note that this will work because we cannot have $F_1 = F_n$ (if $F_1 = F_n$ then x separates G).

CASE II: Every major vertex meets a face that separates J. In [3] we show that in this case there is an interior edge that can be removed producing a smaller circuit graph which by induction can be covered by a 3-tree. The same 3-tree covers $G(J)$. Clearly this argument is not affected by our strengthening of the induction hypothesis.

CASE III: v is a major vertex that does not meet a separating face while all other major vertices meet separating faces.

Let y be a major vertex meeting a separating face F. For every separating face F there are two subpaths of J which are edge disjoint and meet F only at their endpoints. We partially order the set of all these paths associated with all separating faces by inclusion. For the face F, one of the associated subpaths P of J will not contain v. We choose a minimal subpath P' contained in P. The same argument as in [3] shows that the endpoints of P are the vertices of an edge whose removal produces a circuit graph with fewer interior edges and thus yields a covering 3-tree as in Case II. ∎

THEOREM 2: *Every polyhedral map M on the torus has a covering 3-tree.*

Proof: By a theorem of the author [1] there is an annulus A in M consisting of faces of M and containing all vertices of M. Let the two bounding circuits of A be C_1 and C_2. We embed A in the plane so that C_1 bounds a bounded face and C_2 bounds the unbounded face. We place a new vertex V inside C_1 and join it to each vertex of C_1 producing a graph G. Next we place a vertex w in the unbounded face and join it to all vertices of C_2 producing a graph G'.

We claim that G' is 3-connected. To show this it suffices to show that no two faces have a multiply conncected union (clearly all vertices have valence at least three). This condition holds for any two faces of A because it holds in M. No face meeting v will meet a face meeting w, thus the only case remaining is a face F_1 meeting v (or equivalently w) and a face F_2 of A. The face F_1 meets C_1 on an edge e, thus the only way we can have a multiply connected union is if F_2 meets e just at its endpoints. This, however, is impossible because then in M, F_2 would meet a face just at the endpoints of e.

It now follows that $G' - w$ is a circuit graph. By Theorem 1 there is a 3-tree T covering $G' - w$ with v 1-valent. If we remove the edge of T meeting v we have a 3-tree covering the vertices of M. ∎

THEOREM 3: *Every polyhedral map M on the projective plane has a covering 3-tree with any prescribed vertex 1-valent.*

Proof: The complement of a cell in the projective plane is a Möbius strip. Let A be a maximal cell that is the union of faces of A. We shall show that A covers M. First we observe that A can be chosen so that at least one vertex is not on the boundary of A, because we can begin with the set of faces meeting a vertex and take a maximal cell containing that cell.

Suppose now, that a vertex x lies in the complementary Möbius strip S. We choose three independent paths P_1, P_2 and P_3 from x to a vertex of A not on the boundary of A.

For each path P_i let Q_i be the subpath of P_i from x to the first vertex x_i of A on P_i. Two of the three paths, Q_i and Q_j, together with a path P along the boundary of S from x_i to x_j will enclose a cell meeting A on P. Adjoining this cell to A produces a cell that contradicts the maximality of A, thus all vertices of M are in A.

The cell A can be seen to be a circuit graph by embedding it in the plane with the boundary B of A as the unbounded face and joining each vertex of B to a new vertex w. As in the proof of Theorem 2, this new graph is 3-connected, thus A is a circuit graph.

By Theorem 1, a 3-tree covers A with any prescribed vertex 1-valent. The same tree covers M. ∎

THEOREM 4: *Every polyhedral map M on the Klein bottle has a covering 3-tree.*

Proof: By a theorem of the author [1], there is a subset of the faces of M whose union S is a Möbius strip containing all of the vertices of M. We span the boundary of S by a cell. Next we place a new vertex v in the relative interior of this cell and join it to each vertex of the boundeary of S. This produces a polyhedral map M' in the projective plane.

By Theorem 3 we can cover M' with a 3-tree with v being 1-valent. Removing the edge of the 3-tree meeting v gives a 3-tree covering S and thus also covering M. ∎

THEOREM 5: *If M is a polyhedral map on the torus, Klein bottle or projective plane and if M has n vertices then:*

(i) *The vertices of M can be covered by $(n+2)/3$ or fewer disjoint simple paths.*

(ii) *There is a simple path in M with at least $2\log_2 n - 5$ vertices.*

(iii) *There is a simple circuit in M with at least $2\sqrt{2\log_2 n - 5}$ vertices.*

(iv) *Between any two vertices there is a simple path with at least $\sqrt{2\log_2 n - 5}$ vertices.*

Proof: The proof is identical to the proof for planar 3-connected graphs in [3]. ∎

References

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